JOURNAL OF APPROXIMATION THEORY 4, 183–193 (1971)

Best and Near-Best L_1 Approximations by Fourier Series and Chebyshev Series

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Communicated by E. W. Cheney

Received December 10, 1969

Approximations $F_n f$ and $H_n f$ to a function f are defined, respectively, as the partial sums of order n of its expansions in Fourier series and Chebyshev series of the second kind, and they are compared, respectively, with the best trigonometric and best algebraic polynomial approximations $\mathring{f}^B$ and f^B of degree n in $\mathring{L}_1[0, 2\pi]$ and $L_1[-1, 1]$. It is shown that the L_1 norm of $f - F_n f$ differs from that of $f - \mathring{f}^B$ by at most a factor of the order of log n, and that, similarly, the L_1 norm of $f - H_n f$ differs from that of $f - f^B$ by at most a factor of the order of log n. These results are discussed in the context of near-best approximations and minimal projections in L_p spaces. Also, it is shown that, if f has a certain type of lacunary series expansion, then $F_n f$ and $H_n f$ are identical to $\mathring{f}^B$ and f^B, respectively.

1. NEAR-BEST APPROXIMATIONS AND MINIMAL PROJECTIONS

Suppose that f is an element of a normed linear function space X, and that f^* is an element of a subspace Y. Then a practical measure of goodness of the approximation f^* to f can be defined in terms of a concept of "near-best" (see Mason [1]). Specifically, f^* is said to be "near-best within a relative distance ρ" if

$$\|f - f^*\| \leqslant (1 + \rho) \|f - f^B\|, \tag{1}$$

where f^B is any best approximation in Y to f.

One particularly important type of approximation is formed by a projection P of X into a subspace Y. (A projection is a bounded linear map of X into Y, such that $Py = y$ for y in Y). By a standard inequality (Cheney and Price [2]),

$$\|f - Pf\| \leqslant \|I - P\| \cdot \operatorname{dist}(f, Y), \tag{2}$$

where I is the identity map; thus

$$\|f - Pf\| \leqslant \|I - P\| \cdot \|f - f^B\| \tag{3}$$

for any best approximation f^B. But

$$\|I - P\| \leqslant 1 + \|P\|, \tag{4}$$

and therefore

$$\|f - Pf\| \leqslant (1 + \|P\|) \cdot \|f - f^B\|. \tag{5}$$

Thus (5) provides a realization of (1), and any projection P has the property that Pf is a near-best approximation to f within a relative distance $\|P\|$. The infimum of the measure $\|P\|$ taken over all projections P is termed the "relative projection constant," and any projection for which the infimum is attained is called a "minimal projection" [2].

An alternative measure, namely $\|I - P\|$, is suggested by (3), and any projection for which $\|I - P\|$ attains its infimum is called a "cominimal projection" [2].

Two important choices of $\{X, Y\}$ are $\{L_p[a, b], \Pi_n\}$ (approximation by algebraic polynomials of degree n to L_p functions in the L_p norm, $1 \leqslant p \leqslant \infty$) and $\{\mathring{L}_p[0, 2\pi], \mathring{\Pi}_n\}$ (approximation by trigonometric polynomials of degree n to 2π-periodic L_p functions in the L_p norm, $1 \leqslant p \leqslant \infty$). For both of these choices of $\{X, Y\}$, the best approximation f^B in Y to any f in X exists and is unique for each p, $1 \leqslant p \leqslant \infty$ (see [3]). The best approximation is not generally known explicitly, except when $p = 2$, but the minimal and cominimal projections are known for all p in the case of trigonometric approximation. Indeed the Fourier projection F_n, namely the partial sum of order n of the Fourier series of f, is minimal and cominimal in every $\mathring{L}_p[0, 2\pi]$ (Golomb [4, p. 254]). In the particular case of the space $\mathring{C}_{2\pi}$ of continuous periodic functions, F_n is the unique minimal projection (Cheney, et al. [5]), and (see [2]) the relative projection constant is

$$\|F_n\|_\infty = \lambda_n = \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\sin(n + \frac{1}{2})t}{\sin \frac{1}{2}t} \right| dt. \tag{6}$$

The quantity λ_n, the n-th Lebesgue constant, has the asymptotic behaviour (see [2] or [4]):

$$\lambda_n = \frac{4}{\pi^2} \log n + 0(1). \tag{7}$$

The Chebyshev polynomials $T_n(x)$ and $U_n(x)$ of degree n of the first and second kinds are defined by:

$$T_n(x) = \cos n\theta, \qquad U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \tag{8}$$

where $x = \cos\theta$, and the systems $\{T_k\}$ and $U_k\}$ are orthogonal with respect to the weight functions $(1 - x^2)^{-1/2}$ and $(1 - x^2)^{1/2}$, respectively, on $[-1, 1]$. Partial sums of orthogonal expansions are projections; in particular, denoting by $G_n f$ and $H_n f$, respectively, the partial sums of order n of the expansions of f in $\{T_k\}$ and $\{U_k\}$, we have that G_n and H_n are projections. From (6), Golomb [4] immediately deduces that, in $\{C[-1, 1], \Pi_n\}$,

$$\| G_n \|_\infty = \lambda_n . \tag{9}$$

This follows by identifying $\mathring{f}(\theta)$ in $\mathring{C}_{2\pi}$ with $f(x)$ in $C[-1, 1]$ under the transformation $x = \cos\theta$. In this case G_n is not a minimal projection, although the relative projection constant is known to lie in the interval $(\lambda_n - 1)/2 \leqslant x \leqslant \lambda_n$ (see [2]).

In terms of our terminology of "near-best" approximations, (9) establishes that $F_n\mathring{f}$ is near-best within a relative distance λ_n in $\{\mathring{C}_{2\pi}, \mathring{\Pi}_n\}$, and (9) establishes that $G_n f$ is near-best within a relative distance λ_n in $\{C[-1, 1], \Pi_n\}$. The latter result is also proved by a different approach by Powell [6], using properties of orthogonal polynomials. If we denote by $P_n^\omega f$ the partial sum of order n of the expansion of f in polynomials $\{\phi_k\}$ orthonormal with respect to a (nonnegative) weight ω on $[a, b]$, then (see, e.g., [6])

$$\| f - P_n^\omega f \|_\infty \leqslant (1 + \sigma_n) \cdot \| f - f^B \|, \tag{10}$$

where

$$\sigma_n = \max_{a \leqslant x \leqslant b} \int_a^b \omega(y) \left| \sum_{k=0}^n \phi_k(x)\, \phi_k(y) \right| dy. \tag{11}$$

For the particular choice $\{T_k\}$, he shows that

$$\sigma_n = \lambda_n . \tag{12}$$

In Section 2 of the present paper we give results on near-best L_1 approximations by Fourier series and Chebyshev series of the second kind, which are analogous to the above results on near-best L_∞ approximation by Fourier series and Chebyshev series of the first kind. First, we show that

$$\| F_n \|_1 \leqslant \lambda_n , \tag{13}$$

and, hence, F_n is near-best within a relative distance λ_n in $\{\mathring{L}_1[0, 2\pi], \mathring{\Pi}_n\}$.

Next, by making a suitable transformation, we are able to deduce immediately from (13) that

$$\| H_n \|_1 \leqslant \lambda_{n+1}, \tag{14}$$

and, hence, that H_n is near-best within a relative distance λ_{n+1} in $\{L_1[-1, 1], \Pi_n\}$. We also give an alternative proof of the latter result, following the orthogonal polynomial approach of Powell [6]. Specifically, we establish that

$$\| f - P_n^{\omega} f \|_1 \leqslant (1 + \tau_n) \cdot \| f - f^B \|_1, \tag{15}$$

where

$$\tau_n = \max_{a \leqslant y \leqslant b} \omega(y) \int_a^b \left| \sum_{k=0}^{n} \phi_k(x)\, \phi_k(y) \right| dx, \tag{16}$$

and, in the case of $\{U_k\}$, we show that

$$\tau_n \leqslant \lambda_{n+1} \tag{17}$$

and

$$\tau_n = \lambda_{n+1} + 0(1). \tag{18}$$

We remark that we have not obtained equality in any of the relations (13), (14), or (17). Moreover, we have not yet been able to give an explicit characterization of the relative projection constant either in $\{\mathring{L}_1[0, 2\pi], \mathring{\Pi}_n\}$ or in $\{L_1[-1, 1], \Pi_n\}$.

2. Near-Best L_1 Approximations

In this section we retain all the notation of Section 1; in particular, F_n, P_n^{ω}, and H_n denote projections formed by partial sums of Fourier series, orthogonal polynomial series, and Chebyshev series of the second kind, respectively.

Theorem 2.1. $\| F_n \|_1 \leqslant \lambda_n$.

Proof. For any f in $\mathring{L}_1[0, 2\pi]$, Dirichlet's formula holds (compare [7, p. 120]):

$$(F_n f)(\theta) = \frac{1}{2\pi} \int_0^{2\pi} f(t + \theta)\, D_n(t)\, dt, \tag{19}$$

where

$$D_n(t) = \frac{\sin(n + \frac{1}{2})t}{\sin \frac{1}{2}t}. \tag{20}$$

From (19),

$$\|(F_n f)(\theta)\|_1 = \frac{1}{2\pi} \int_0^{2\pi} \left| \int_0^{2\pi} f(t+\theta)\, D_n(t)\, dt \right| d\theta$$

$$\leqslant \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} | f(t+\theta)| \cdot | D_n(t)|\, dt\, d\theta. \tag{21}$$

Reversing the order of integrations in (21) (by Fubini's theorem), using the periodicity of f, and recalling the definition (6) of λ_n, we obtain

$$\|(F_n f)(\theta)\|_1 \leqslant \lambda_n \cdot \|f(\theta)\|_1, \tag{22}$$

and the required result follows. Q.E.D.

From 2.1 and (5) we deduce:

COROLLARY 2.2. $\|f - F_n f\|_1 \leqslant (1 + \lambda_n) \cdot \|f - f^B\|_1$.

COROLLARY 2.3. $\| H_n \|_1 \leqslant \lambda_{n+1}$.

Proof of 2.3. For any f in $L_1[-1, 1]$, define $f^0 \in \mathring{L}_1[0, 2\pi]$ by

$$f^0(\theta) = \sin\theta \cdot f(\cos\theta). \tag{23}$$

From the definition (8) of $\{U_k\}$ it follows that

$$F_{n+1} f^0 = (H_n f)^0. \tag{24}$$

Since f^0 is odd and periodic,

$$\| f^0 \|_1 = 2 \int_0^{\pi} | f^0(\theta)|\, d\theta = 2 \int_0^{\pi} | f(x)| \cdot | \sin\theta \,|\, d\theta$$

$$= 2 \int_0^{\pi} | f(x)| \sin\theta\, d\theta = 2 \int_{-1}^{1} | f(x)|\, dx$$

$$= 2 \| f\|_1. \tag{25}$$

Similarly, from (24),

$$\| F_{n+1} f^0 \|_1 = 2 \| H_n f\|_1. \tag{26}$$

Hence the required result follows by substituting (25) and (26) into (22). Q.E.D.

From 2.3 and (5) we deduce:

COROLLARY 2.4. $\|f - H_n f\|_1 \leqslant (1 + \lambda_{n+1}) \cdot \|f - f^B\|_1$.

Corollary 2.4 may alternatively be obtained from first principles by Powell's approach [6] as follows:

THEOREM 2.5. $\|f - P_n^{\omega} f\|_1 \leqslant (1 + \tau_n) \cdot \|f - f^B\|_1$. (27)

Proof. Set $\epsilon \equiv f - P_n^{\omega} f$ and $\epsilon^B \equiv f - f^B$. Then (compare [6])

$$\epsilon(x) = \epsilon^B(x) - \int_a^b \epsilon^B(y)\, \omega(y) \sum_{k=0}^{n} \phi_k(x)\, \phi_k(y)\, dy.$$

Taking moduli, applying standard inequalities, and integrating,

$$\|\epsilon\|_1 \leqslant \|\epsilon^B\|_1 + \int_a^b \int_a^b |\epsilon^B(y)| \cdot \omega(y) \cdot \left| \sum_{k=0}^{n} \phi_k(x)\, \phi_k(y) \right| dy\, dx.$$

Reversing the order of integrations (by Fubini's theorem) and applying Hölder's inequality, we obtain (27).

LEMMA 2.6. *For the orthonormal system*

$$\{\phi_k\} \equiv \left\{ \sqrt{\frac{2}{\pi}}\, U_k \right\},$$

$$\tau_n \leqslant \lambda_{n+1} \qquad \textit{and} \qquad \tau_n = \lambda_{n+1} + 0(1).$$

Proof. In this case, $\omega(x) \equiv \sqrt{(1 - x^2)}$. Setting $x = \cos\theta$ and $y = \cos\psi$ in (16),

$$\tau_n = \max_{0 \leqslant \psi \leqslant \pi} \frac{2}{\pi} \int_0^{\pi} \left| \sum_{k=0}^{n+1} \sin k\theta \sin k\psi \right| d\theta, \tag{29}$$

$$= \max_{\psi} \frac{1}{\pi} \int_0^{\pi} \left| \sum_{k=0}^{n+1} \cos k(\theta - \psi) - \sum_{k=0}^{n+1} \cos k(\theta + \psi) \right| d\theta,$$

$$\leqslant \max_{\psi} \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} [|D_{n+1}(\theta - \psi)| + |D_{n+1}(\theta + \psi)|]\, d\theta, \tag{30}$$

where D_{n+1} is defined by (20). The integral in (30) is independent of ψ, and hence, setting $\psi = 0$,

$$\tau_n \leqslant \lambda_{n+1}. \tag{31}$$

Also, setting $\psi = \pi/2$ in the integrand in (29),

$$\tau_n \geqslant \frac{2}{\pi} \int_0^{\pi} \left| \sum_{k=1}^{n+1} \sin k\theta \sin \frac{k\pi}{2} \right| d\theta. \tag{32}$$

By summing the series in (32) and applying analysis similar to that on p. 213 of [7], it is straightforward to show that

$$\tau_n \geqslant \frac{4}{\pi^2} \log \frac{1}{2} (n + 1). \tag{33}$$

The required results follow from (7), (31), and (33). Q.E.D.

Clearly, Corollary 2.4 is an immediate consequence of 2.5 and 2.6.

3. Best Approximations and Lacunary Series

A basic property of the Chebyshev polynomials (see [1]) is that $2^{1-n}T_n(x)$ and $2^{-n}U_n(x)$ are the best monic polynomial approximations of degree n to the zero function in $L_\infty[-1, 1]$ and $L_1[-1, 1]$, respectively. It immediately follows that, if f is a polynomial of degree $n + 1$, then the partial sums of order n of its $\{T_k\}$ and $\{U_k\}$ expansions are, respectively, its best L_∞ and L_1 polynomial approximations of degree n. Similar, but more general results hold for various functions f having lacunary series expansions (i.e., expansions in which "almost all" coefficients are zero). For example, if b is any odd integer greater than one and a is any real number of absolute value less than one, then the function

$$f = \sum_{i=0}^{\infty} a^i T_{b^i}(x) \tag{34}$$

is in $C[-1, 1]$ and has the property (p. 132 of [7]) that, for every n, the partial sum of order n of its $\{T_k\}$ expansion is precisely its best L_∞ polynomial approximation of degree n. The function

$$\mathring{f} = \sum_{i=0}^{\infty} a^i \cos b^i\theta \tag{35}$$

is related under the transformation $x = \cos\theta$ to the function f of (34) by the identity

$$\mathring{f}(\theta) = f(\cos\theta) = f(x).$$

Hence $\|\mathring{f}(\theta)\|_\infty = \|f(x)\|_\infty$ and $\|(F_n\mathring{f})(\theta)\|_\infty = \|(G_nf)(x)\|_\infty$, and we deduce that $\mathring{f}$ has the property that the partial sum of order n of its Fourier series is precisely its best L_∞ even-trigonometric approximation of degree n.

It is interesting that $\mathring{f}$, as defined in (35), is in fact the Weierstrass function ([7, p. 128]) which, for $|ab| > 1$, is nowhere differentiable. However, for $|ab| < 1$, (35) may be differentiated term by term to yield a uniformly convergent expansion.

We now prove some analogous results in the L_1 norm for lacunary series and Chebyshev series of the second kind.

Consider a real function f having an expansion

$$f \sim \sum_{i=0}^{\infty} c_i U_{k_i}(x) \tag{36}$$

in which each c_i is nonzero and $\{k_i\}$ is a sequence of integers satisfying

$$\begin{gathered} k_0 = 0, \qquad k_1 > 0, \\ (k_{i+1} + 1) = r_i(k_i + 1) \qquad (i \geqslant 1), \end{gathered} \tag{37}$$

where each r_i is an integer greater than one. For every n we denote the best L_1 polynomial approximation of degree n to f by $f_n{}^B$. Now, using the definition of U_k and the transformation $x = \cos\theta$, we obtain from (37) that

$$\begin{aligned} U_{k_{i+1}}(x) &= \frac{\sin(k_{i+1} + 1)\theta}{\sin\theta} \\ &= \frac{\sin(k_i + 1)\theta}{\sin\theta} \cdot \frac{\sin r_i(k_i + 1)\theta}{\sin(k_i + 1)\theta}. \end{aligned}$$

Thus

$$U_{k_{i+1}}(x) = U_{k_i}(x) \cdot R_i(x), \tag{38}$$

where

$$R_i(x) = \frac{\sin r_i(k_i + 1)\theta}{\sin(k_i + 1)\theta} \qquad \text{for} \quad x = \cos\theta.$$

Clearly R_i is a polynomial in x of degree $(r_i - 1)(k_i + 1)$. Also

$$R_i(x) = \frac{\sin r_i\psi}{\sin\psi}, \qquad \text{where} \quad \psi = (k_i + 1)\theta,$$

and hence

$$|R_i(x)| \leqslant r_i \qquad \text{for all} \quad x \text{ in } [-1, 1]. \tag{39}$$

If, for any given m, we define

$$M = k_{m+1} - 1, \tag{40}$$

then it is clear from (36) that $H_N f$ is identical to $H_M f$ for every N such that

$$k_m \leqslant N < k_{m+1}. \tag{41}$$

Now, if f is continuous, a sufficient condition for $H_M f$ to be the best L_1

approximation $f_M{}^B$ is that $f - H_M f$ should have precisely $M + 1$ changes of sign occurring at the $M + 1$ zeros of U_{M+1} (see [1] or [3]). Thus, if we can show that

(i) f is continuous, and

(ii) for each m,

$$f - H_M f = U_{M+1}(x) \cdot \Phi_m(x), \tag{42}$$

where Φ_m does not change sign on $[-1, 1]$, then it will follow that $H_M f$ is identical with $f_M{}^B$. It will then further follow that, for every N satisfying (41), $H_N f$ is identically $f_N{}^B$, and hence that $H_n f$ is identically $f_n{}^B$ for all n. In the following theorem and corollary we give restrictions on $\{c_i\}$ and $\{r_i\}$ which are sufficient to ensure the two required properties.

THEOREM 3.1. *If $\{c_i\}$ is bounded and if, for every m,*

$$s_m = \sum_{i=m+2}^{\infty} | c_i | \cdot \left\{ \prod_{j=m+2}^{i} r_j \right\} \leqslant | c_{m+1} |, \tag{43}$$

then the series (36) *is uniformly convergent, f is in $C[-1, 1]$, and $H_n f$ is identically $f_n{}^B$ for every n.*

Proof. Since $| U_{k_i} |$ is bounded by $k_i + 1$, the expansion (36) is majorized by

$$| c_0 | + | c_1 | \cdot (k_1 + 1) + \sum_{i=2}^{\infty} | c_i | \cdot (k_i + 1)$$

$$= | c_0 | + | c_1 | \cdot (k_1 + 1) + s_0(k_2 + 1), \qquad \text{by (37) and (43).}$$

Thus (36) is uniformly convergent and, hence, f is in $C[-1, 1]$. Now, from (38),

$$f - H_M f = \sum_{i=m+1}^{\infty} c_i U_{k_i}(x) = U_{M+1}(x) \cdot \Phi_m(x),$$

where

$$\Phi_m = c_{m+1} + \sum_{i=m+2}^{\infty} c_i \left\{ \prod_{j=m+2}^{i} R_j(x) \right\}.$$

By (39),

$$| \Phi_m | \leqslant | c_{m+1} | + | s_m |$$

and, hence, Φ_m is in $C[-1, 1]$. By (43), Φ_m does not change sign in $[-1, 1]$.

COROLLARY 3.2. *If f has the expansion*

$$f \sim \sum_{i=0}^{\infty} a^i U_{b^i-1}(x), \tag{44}$$

where b is any integer greater than one and a is any real number such that

$$0 < | ab | \leqslant \tfrac{1}{2} \tag{45}$$

then the series (44) *is uniformly convergent, f is in $C[-1, 1]$, and $H_n f$ is identically $f_n{}^B$ for every n.*

Proof. In this case,

$$c_i = a^i \qquad \text{and} \qquad r_i = b. \tag{46}$$

Hence

$$\begin{aligned} s_m &= \sum_{i=m+2}^{\infty} | c_i | \cdot \left\{ \prod_{j=m+2}^{\infty} r_j \right\}, \\ &= \sum_{i=m+2}^{\infty} | a |^i \, b^{i-m-1} = \frac{| ab |}{1 - | ab |} \cdot | a |^{m+1}, \\ &\leqslant | a |^{m+1} = | c_{m+1} |, \qquad \text{by (45)}. \end{aligned}$$

Since $| a | < 1$, s_m is convergent and the result follows immediately from 3.1. Q.E.D.

The function

$$f^0(\theta) = \sum_{i=0}^{\infty} c_i \sin(k_i + 1)\theta$$

is related to f of (36) by

$$f^0(\theta) = \sin \theta \cdot f(\cos \theta).$$

Hence, as in the proof of 2.3 above,

$$\| f^0 \|_1 = 2 \| f \|_1 \qquad \text{and} \qquad \| F_{n+1} f^0 \|_1 = 2 \| H_n f \|_1 ,$$

and we deduce that, when $\{c_i\}$ satisfies the hypotheses of Theorem 3.1, the partial sum $F_n f^0$ of the Fourier series of f^0 is precisely the best L_1 odd-trigonometric approximation of order n to f^0.

ACKNOWLEDGMENTS

This research work was supported in part by the National Research Council of Canada. We would like to thank Professor E. W. Cheney and the referee for some most constructive suggestions.

REFERENCES

1. J. C. MASON, Orthogonal polynomial approximation methods in numerical analysis, *in* "Approximation Theory," (Proceedings of a Symposium), (A. Talbot, Ed.,) Academic Press, 1970.
2. E. W. CHENEY AND K. H. PRICE, Minimal projections, *in* "Approximation Theory," (Proceedings of a Symposium), (A. Talbot, Ed.), Academic Press, 1970.
3. J. R. RICE, "The Approximation of Functions," Vol. I, Addison-Wesley, Reading, Mass., 1964.
4. M. GOLOMB, "Lectures on Theory of Approximation," Argonne National Laboratory, Argonne, Ill., 1962.
5. E. W. CHENEY, C. HOBBY, P. D. MORRIS, F. SCHURER, AND D. E. WULBERT, On the minimal property of the fourier projection, *Bull. Amer. Math. Soc.* **75** (1969), 51–52.
6. M. J. D. POWELL, On the maximum errors of polynomial approximations defined by interpolation and least-squares criteria, *Comput. J.* **9** (1967), 404–407.
7. E. W. CHENEY, "Introduction to Approximation Theory." McGraw-Hill, New York, 1966.